INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 9, 185–191 (1989)

EFFICIENT QUADRATURE FOR A BOUNDARY ELEMENT METHOD TO COMPUTE THREE-DIMENSIONAL STOKES FLOW

WERNER VARNHORN

Fachbereich Mathematik-Informatik, Universität GH Paderborn, D-4790 Paderborn, West Germany

SUMMARY

A collocation-type boundary element method based on bilinear B-splines is used for the numerical solution of the Stokes Dirichlet problem in bounded domains $D \subset R^3$. The computation of the influence matrix requires the numerical evaluation of weakly singular integrals on the domain boundary if the usual double-layer potential *ansatz* is chosen. Here mostly standard methods with disjoint grids for collocation and integration are used. We develop a special integration scheme based on triangular co-ordinates near the singularity and show its efficiency compared with the method mentioned above.

KEY WORDS Stokes equations Boundary elements Quadrature Triangular co-ordinates

INTRODUCTION

During the last 15 years, boundary element methods have been established more and more for the numerical treatment of boundary value problems in many engineering fields.[1–3] These methods can be used if a fundamental solution for the differential equations is explicitly known. Then the solution of the boundary value problem considered in a certain domain can be represented by boundary layer potentials, whose unknown densities must be determined as the solution of integral equations over the domain boundary. Hence the 'dimension' of the problem is reduced.

The most popular method to discretize the boundary integral equations is the collocation procedure: replacing the unknown densities by functions depending on a finite number M of unknowns, the corresponding integral equations are required to hold on M chosen boundary points only. The result is a system of linear algebraic equations with a non-sparse influence matrix. Its computation requires the numerical quadrature of integrals with singular integrands for each matrix element, which makes this part of the procedure costly.

In the present paper we treat the interior Stokes boundary value problem with a collocation method developed in Reference 4. By a special choice of suitable quadrature formulae, we can improve the accuracy and, at the same time, reduce the costs (CPU) essentially. Although the methods developed here are presented only for concrete model problems in fluid dynamics (flow inside a unit ball), they can be easily applied to many other problems in various fields.

0271-2091/89/020185–07$05.00

Received 27 March 1987
Revised 18 March 1988

HYDRODYNAMICAL POTENTIAL THEORY

Let us consider the Stokes equations

$$\begin{aligned} -\Delta \mathbf{v} + \nabla p &= 0 \quad \text{in } G \\ \nabla \cdot \mathbf{v} &= 0 \quad \text{in } G \\ \mathbf{v} &= \mathbf{b} \quad \text{on } \partial G \end{aligned} \tag{1}$$

in a bounded domain $G \subset R^3$ with a smooth boundary ∂G. These equations describe the motion of a viscous incompressible fluid contained in G: $\mathbf{v} := (v_1, v_2, v_3)$ represents the velocity of a particle of fluid and p the pressure. For the given boundary value $\mathbf{b} \in C(\partial G)$ we always assume the compatibility condition

$$\int_{\partial G} \mathbf{b}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \, do_y = 0, \tag{2}$$

where $\mathbf{n}(\mathbf{y})$ denotes the outward unit normal in $\mathbf{y} \in \partial G$.

For the solution $\mathbf{v}$, p of (1) let us use the classical *ansatz* of hydrodynamical double-layer potentials,[5, 6] which for $\mathbf{x} \in G$ and $\boldsymbol{\Phi} \in C(\partial G)$ are of the form

$$\mathbf{V}(\mathbf{x}) := D\boldsymbol{\Phi}(\mathbf{x}) := \int_{\partial G} \mathbf{D}(\mathbf{x}, \mathbf{y}) \boldsymbol{\Phi}(\mathbf{y}) \, do_y, \tag{3}$$

$$P(\mathbf{x}) := d\boldsymbol{\Phi}(\mathbf{x}) := \int_{\partial G} \mathbf{d}(\mathbf{x}, \mathbf{y}) \cdot \boldsymbol{\Phi}(\mathbf{y}) \, do_y. \tag{4}$$

Here the matrix $\mathbf{D}(\mathbf{x}, \mathbf{y})$ and the vector $\mathbf{d}(\mathbf{x}, \mathbf{y})$ for $\mathbf{x} \in G$ and $\mathbf{y} \in \partial G$ are defined by

$$\mathbf{D}(\mathbf{x}, \mathbf{y}) := \left(-\frac{3\mathbf{r} \cdot \mathbf{n} r_i r_j}{4\pi |\mathbf{r}|^5} \right)_{i,j=1,2,3}, \tag{5}$$

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) := \left(-\frac{3\mathbf{r} \cdot \mathbf{n} r_i - \mathbf{r}^2 \mathbf{n}_i}{2\pi |\mathbf{r}|^5} \right)_{i=1,2,3}, \tag{6}$$

where we set $\mathbf{r} := \mathbf{x} - \mathbf{y}$, $\mathbf{n} := \mathbf{n}(\mathbf{y})$ for abbreviation and $|\ |$ for the Euclidean norm.[5, 6] Because $\mathbf{V}, P$ satisfy the first two identities in (1), it remains to determine the unknown density $\boldsymbol{\Phi}$ such that the boundary condition is fulfilled also. Owing to the jump relations of (3), if $\boldsymbol{\Phi} \in C(\partial G)$, this leads to the boundary integral equation system

$$(\tfrac{1}{2} I + D)\boldsymbol{\Phi} = \mathbf{b} \quad \text{on } \partial G, \tag{7}$$

which is a Fredholm system of the second kind.[5, 6] Here we used $I := (\delta_{ij})_{i,j=1,2,3}$, with δ_{ij} as the Kronecker symbol, and $D\boldsymbol{\Phi}$ for the direct value of (3) on ∂G.

It is known that there exists a solution $\boldsymbol{\Phi} \in C(\partial G)$ of (7), but it is not uniquely determined.[5, 6] Because for numerical purposes a uniquely solvable system of integral equations is required, we replace (7) by

$$(\tfrac{1}{2} I + D - N)\boldsymbol{\Phi} = \mathbf{b} \quad \text{on } \partial G \tag{8}$$

with

$$N\boldsymbol{\Phi}(\mathbf{x}) := \mathbf{n}(\mathbf{x}) \int_{\partial G} \mathbf{n}(\mathbf{y}) \cdot \boldsymbol{\Phi}(\mathbf{y}) \, do_y. \tag{9}$$

It can be shown[7] that (8) is uniquely solvable and that its solution $\Phi \in C(\partial G)$ satisfies (7) if (2) holds.

Finally let us use a smoothing property of the double-layer potential (3) to reduce round-off errors in the computations later on. Due to Gauss's formula[5,6]

$$D\mathbf{c}(\mathbf{x}) := \int_{\partial G} \mathbf{D}(\mathbf{x}, \mathbf{y})\mathbf{c}\, do_y = \begin{cases} 0 & \\ \frac{1}{2}\mathbf{c} & \text{if} \\ \mathbf{c} & \end{cases} \begin{cases} \mathbf{x} \notin \bar{G} \\ \mathbf{x} \in \partial G \\ \mathbf{x} \in G \end{cases}, \tag{10}$$

which holds for any constant vector $\mathbf{c} \in R^3$, the system (8) is equivalent to

$$(I + D^* - N)\Phi = \mathbf{b} \quad \text{on } \partial G \tag{11}$$

with

$$D^*\Phi(\mathbf{x}) := \int_{\partial G} \mathbf{D}(\mathbf{x}, \mathbf{y})(\Phi(\mathbf{y}) - \Phi(\mathbf{x}))\, do_y.$$

This system will be considered in the following. Its solution Φ determines the solution of (1) by (3), (4), where again for numerical reasons

$$\mathbf{V}(\mathbf{x}) = \Phi(\mathbf{z}) + \int_{\partial G} \mathbf{D}(\mathbf{x}, \mathbf{y})(\Phi(\mathbf{y}) - \Phi(\mathbf{z}))\, do_y \tag{12}$$

is used instead of (3) if $\mathbf{x} \in G$ is situated near its projection $\mathbf{z} \in \partial G$.

A COLLOCATION METHOD

For illustration let us restrict our considerations to the Stokes equations (1) in the unit ball $B \subset R^3$. Then we can represent its boundary $S := \partial B$ by one global co-ordinate system only. Setting $\bar{S} := [0, \frac{1}{2}] \times [0, 1]$, we use

$$\bar{\mathbf{p}}: \begin{cases} \bar{S} \longrightarrow S \\ \bar{x} := (\bar{x}_1, \bar{x}_2) \longrightarrow \mathbf{x} = (x_1, x_2, x_3) \end{cases} \tag{13}$$

with normed polar co-ordinates

$$x_1 := \sin(2\pi\bar{x}_1)\cos(2\pi\bar{x}_2), \qquad x_2 := \sin(2\pi\bar{x}_1)\sin(2\pi\bar{x}_2), \qquad x_3 := \cos(2\pi\bar{x}_1).$$

The Jacobian of this transformation is given by

$$\bar{J}(\bar{x}) := 4\pi^2 \sin(2\pi\bar{x}_1).$$

By (13), the system (11) is transformed into the integral equations

$$(I + \bar{D} - \bar{N})\bar{\Phi} = \bar{\mathbf{b}} \quad \text{on } \bar{S}, \tag{14}$$

where here and in the following the overbar indicates $\bar{S}$ as the domain of definition. Thus for example in our case of the unit sphere, (9) leads to

$$\bar{N}\bar{\Phi}(\bar{\mathbf{x}}) = \bar{\mathbf{p}}(\bar{\mathbf{x}}) \int_{\bar{S}} \bar{\mathbf{p}}(\bar{\mathbf{y}}) \cdot \bar{\Phi}(\bar{\mathbf{y}}) \bar{J}(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}.$$

For the discretization of (14) we use a collocation procedure from Reference 4 in a slightly modified version. Let $\bar{S}$ be divided into $2N^2$ squares $\bar{Q}$ of length $h := 1/2N$, whose corner points

$$\bar{C}_h := \{\bar{\mathbf{x}} = (ih, jh) | i = 0, 1, \ldots, N; j = 0, 1, \ldots, 2N\} \tag{15}$$

define on $\bar{S}$ a collocation grid. With

$$w(t) := \begin{cases} 1 + t & (-1 \le t \le 0) \\ 1 - t & (0 \le t \le 1) \\ 0 & \text{otherwise} \end{cases},$$

for every collocation point $\bar{\mathbf{x}} \in \bar{C}_h$ we fix a bilinear B-spline $\bar{w}_{\bar{x}}: \bar{S} \to \mathbf{R}$ by

$$\bar{w}_{\bar{x}}(\bar{\mathbf{y}}) := w((\bar{y}_1 - \bar{x}_1)/h)\, w((\bar{y}_2 - \bar{x}_2)/h) \tag{16}$$

and look for an approximate solution $\bar{\boldsymbol{\Phi}}_h$ of (14) in the form

$$\bar{\boldsymbol{\Phi}}_h(\bar{\mathbf{y}}) := \sum_{\bar{x} \in \bar{C}_h} \mathbf{a}^h_{\bar{x}} \bar{w}_{\bar{x}}(\bar{\mathbf{y}}). \tag{17}$$

Here the unknown coefficients $\mathbf{a}^h_{\bar{x}} = \bar{\boldsymbol{\Phi}}_h(\bar{\mathbf{x}}) \in R^3$ represent an approximation for the vectors $\bar{\boldsymbol{\Phi}}(\bar{\mathbf{x}})$, $\bar{\mathbf{x}} \in \bar{C}_h$, and can be determined by the collocation equations

$$(I + \bar{D}^* - \bar{N})\bar{\boldsymbol{\Phi}}_h = \bar{\mathbf{b}} \quad \text{on } \bar{C}_h. \tag{18}$$

This is a system of linear algebraic equations containing a non-sparse matrix of $3(N+1)(2N+1)$ degrees of freedom. It can be solved on a computer if, as in the next section, suitable quadrature formulae for the remaining integrals are chosen.

NUMERICAL INTEGRATION

A common method for the numerical quadrature of the weakly singular integrals in (18) is the midpoint rule.[4] Here we define on $\bar{S}$ an integration grid $\bar{I}_h$ by

$$\bar{I}_h := \{\bar{\mathbf{y}} = ((i - \tfrac{1}{2})h, (j - \tfrac{1}{2})h) | i = 1, 2, \ldots, N; j = 1, 2, \ldots, 2N\}$$

and replace for every $\bar{\mathbf{x}} \in \bar{C}_h$

$$\bar{D}^* \bar{\boldsymbol{\Phi}}_h(\bar{\mathbf{x}}) = \int_{\bar{S}} \bar{\mathbf{D}}(\bar{\mathbf{x}}, \bar{\mathbf{y}})(\bar{\boldsymbol{\Phi}}_h(\bar{\mathbf{y}}) - \bar{\boldsymbol{\Phi}}_h(\bar{\mathbf{x}}))\bar{J}(\bar{\mathbf{y}})\, d\bar{\mathbf{y}}$$

by

$$\bar{D}^*_h \bar{\boldsymbol{\Phi}}_h(\bar{\mathbf{x}}) := h^2 \sum_{\bar{\mathbf{y}} \in \bar{I}_h} \bar{\mathbf{D}}(\bar{\mathbf{x}}, \bar{\mathbf{y}})(\bar{\boldsymbol{\Phi}}_h(\bar{\mathbf{y}}) - \bar{\boldsymbol{\Phi}}_h(\bar{\mathbf{x}}))\bar{J}(\bar{\mathbf{y}}).$$

Thus using (17) to obtain

$$\bar{\boldsymbol{\Phi}}_h(\bar{\mathbf{y}}) = \tfrac{1}{4} \sum_{\substack{\bar{z} \in \bar{C}_h \\ |\bar{z} - \bar{\mathbf{y}}| \le h}} \mathbf{a}^h_{\bar{z}}, \qquad \bar{\boldsymbol{\Phi}}_h(\bar{\mathbf{x}}) = \mathbf{a}^h_{\bar{x}},$$

after an analogue treatment of the integrals $\bar{N}\bar{\boldsymbol{\Phi}}_h(\bar{\mathbf{x}})$, the resulting algebraic system for the unknowns $\mathbf{a}^h_{\bar{x}}(\bar{\mathbf{x}} \in \bar{C}_h)$ is ready for implementation. Because here the grids for collocation and integration are disjoint, it is not necessary to smooth out the singularity. On the other hand, the computation of almost every element in the influence matrix requires the computation of the double-layer kernel (5) four times, so that this might be too expensive.

We therefore propose another quadrature formula using triangular co-ordinates[8] together with the trapezoidal rule. Choosing the same grids for collocation and integration, let us first replace

the regular integrals $\bar{N}\bar{\Phi}_h(\bar{\mathbf{x}})$ in (18) by

$$\bar{N}_h\bar{\Phi}_h(\mathbf{x}) := \bar{\mathbf{p}}(\bar{\mathbf{x}})(h^2/4)\sum_{\bar{z}\in\bar{C}_h}\beta_{\bar{z}}\bar{\mathbf{p}}(\bar{\mathbf{z}})\cdot\mathbf{a}_{\bar{z}}^h\bar{J}(\bar{\mathbf{z}}),$$

where the trapezoidal weights $\beta_{\bar{z}}\in\{1, 2, 4\}$ in $\bar{z}\in\bar{C}_h$ are given as usual:

1	2	2	2	2	2	.	.	2	1
2	4	4	4	4	4	.	.	4	2
2	4	4	4	4	4	.	.	4	2
2	4	4	4	4	4	.	.	4	2
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
2	4	4	4	4	4	.	.	4	2
1	2	2	2	2	2	.	.	2	1

To approximate the weakly singular integrals $\bar{D}^*\bar{\Phi}_h(\bar{\mathbf{x}})$, we use a transformation on triangular co-ordinates in all squares $\bar{Q}$ containing the singularity $\bar{\mathbf{x}}$ in one of its corner points. To do so, let us for short change the notation and consider the integral

$$K := \int_0^h\int_0^h k(u, w)\,\mathrm{d}w\mathrm{d}u$$

with a function k having a weak singularity for $u = w = 0$. Due to

$$K = \int_0^h\int_0^u (k(u, w) + k(w, u))\,\mathrm{d}w\mathrm{d}u,$$

with the help of the transformation $u = s$, $w = st/h$ we obtain

$$K = (1/h)\int_0^h\int_0^h (k(s, st/h) + k(st/h, s))s\,\mathrm{d}s\mathrm{d}t,$$

which leads to

$$K \approx (h^2/4)(k(h, 0) + k(0, h) + 2k(h, h))$$

if the trapezoidal rule is used. Thus for the remaining integrals $\bar{D}^*\bar{\Phi}_h(\bar{\mathbf{x}})$, $\bar{\mathbf{x}}\in\bar{C}_h$, our quadrature formula is chosen to be

$$\bar{D}_h^*\bar{\Phi}_h(\bar{\mathbf{x}}) := (h^2/4)\sum_{\bar{z}\in\bar{C}_h}\beta_{\bar{z},\bar{x}}\bar{\mathbf{D}}(\bar{\mathbf{x}}, \bar{\mathbf{z}})(\mathbf{a}_{\bar{z}}^h - \mathbf{a}_{\bar{x}}^h)\bar{J}(\bar{\mathbf{z}})$$

with modified trapezoidal weights $\beta_{\bar{z},\bar{x}}\in\{0, 1, \ldots, 5\}$ depending additionally on the position of the singularity $\bar{\mathbf{x}}$. For example, we use

1	3	2	3	2	.	.	2	1	
2	4	0	4	4	.	.	4	2	
2	5	4	5	4	.	.	4	2	
2	4	4	4	4	.	.	4	2	
.	.	.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.	.	
2	4	4	4	4	.	.	4	2	
1	2	2	2	2	.	.	2	1	

or

1	2	2	2	.	.	2	2	1	
2	5	4	4	.	.	4	5	2	
0	4	4	4	.	.	4	4	0	
2	5	4	4	.	.	4	5	2	
2	4	4	4	.	.	4	4	2	
.	.	.	.	.	.	.	.	.	
2	4	4	4	.	.	4	4	2	
1	2	2	2	.	.	2	2	1	

where the weight 0 indicates the position of the singularity. Now also for the weakly singular integrals the grids for collocation and integration may coincide, which, in contrast to the midpoint rule, requires the computation of the double-layer kernel once per matrix element only. This is the main reason for the reduction of the CPU time to almost one-quarter (see examples below). Higher-order quadrature schemes, like the Gaussian for example, are even more costly than the midpoint rule and not so efficient in the present case, where linear splines are used for interpolation only.

An approximate solution $\mathbf{v}^h$, p^h of the considered Stokes system can now be obtained from (3) ((12) resp.) and (4) by using the same quadrature formulae as for the corresponding boundary integral equations. In the last section some test calculations are carried out to compare accuracy and computing time for both the midpoint and the modified trapezoidal schemes.

NUMERICAL TEST

The model problems considered here have no physical meaning, but, because their solutions are explicitly known, the accuracy of the quadrature schemes used can be easily assessed. Although the flow region is the unit ball $B \subset R^3$ with boundary $S := \partial B$, the computations are carried out without using any symmetry properties of the ball. The resulting influence matrix with 975 degrees of freedom—we use the same spatial step size ($N = 12$) in all examples—is strongly diagonal-dominant owing to (10), hence well conditioned, and has been inverted directly. For each component ($k = 1, 2, 3$) the mean relative error (%)

$$E_k := (100/L) \sum_{j=1}^{L} |(v_k(\mathbf{z}^j) - v_k^h(\mathbf{z}^j))/v_k(\mathbf{z}^j)|, \quad \mathbf{z}^j \in B$$

between the exact solution v_k and the computed solution v_k^h together with the computing time (CPU) are presented below. The results show the efficiency of the trapezoidal rule as expected.

Example 1

The functions $\mathbf{v}: R^3 \to R^3$, $p: R^3 \to R$ defined by

$$\mathbf{v}(\mathbf{x}) := \begin{pmatrix} x_3 - x_2 \\ x_1 - x_3 \\ x_2 - x_1 \end{pmatrix}, \qquad p(\mathbf{x}) := \text{constant}$$

represent the solution of a Stokes model problem (1) in B with prescribed boundary values $\mathbf{b} := \mathbf{v}$ on S. In this case for the error E_k(%) and the CPU percentage (midpoint rule $\approx$ 100%) we obtain the following result:

Rule	E_1	E_2	E_3	CPU
Midpoint	1·046	1·042	1·801	100%
Trapezoidal	0·463	0·462	1·300	23%

Example 2

Consider a Leray-type solution (Reference 5, p. 138) defined by

$$\mathbf{v}(\mathbf{x}) := \begin{pmatrix} x_1^2 + 3 - 2|\mathbf{x}|^2 \\ x_1 x_2 \\ x_1 x_3 \end{pmatrix}, \qquad p(\mathbf{x}) := -10x_1 + \text{constant}$$

This solves (1) with prescribed boundary values $\mathbf{b} := \mathbf{v}$ on S. Here we obtain:

Rule	E_1	E_2	E_3	CPU
Midpoint	2·059	7·237	2·076	100%
Trapezoidal	0·254	2·051	1·175	27%

Example 3

Using the fundamental tensor of the Stokes system (Reference 5, p. 51) in a fixed point $\mathbf{z} := (z_1, 0, 0)$ outside the unit ball B (choose $z_1 > 1$!), we obtain for example

$$\mathbf{v}(\mathbf{x}) := |\mathbf{x} - \mathbf{z}|^{-3} \begin{pmatrix} |\mathbf{x} - \mathbf{z}|^2 + (x_1 - z_1)^2 \\ (x_1 - z_1)x_2 \\ (x_1 - z_1)x_3 \end{pmatrix}, \qquad p(x) := 2(x_1 - z_1)|\mathbf{x} - \mathbf{z}|^{-3}.$$

These functions solve (1) in B with prescribed boundary values $\mathbf{b} := \mathbf{v}$ on S. For $z_1 = 1{\cdot}1$ we quote:

Rule	E_1	E_2	E_3	CPU
Midpoint	5·461	11·199	14·289	100%
Trapezoidal	1·862	4·702	4·350	29%

ACKNOWLEDGEMENT

This work was supported by Deutsche Forschungsgemeinschaft, Bonn.

REFERENCES

1. C. A. Brebbia, J. C. F. Telles and L. C. Wrobel, *Boundary Element Techniques*, Springer-Verlag, Berlin, 1984.
2. G. Kuhn and W. Möhrmann, 'Boundary element method in elastostatics: theory and application', *Appl. Math. Modelling*, **7**, 97–105 (1983).
3. J. Mackerle and T. Anderson, 'Boundary element software in engineering', *Adv. Eng. Software*, **6**, 66–102 (1986).
4. F. K. Hebeker, 'Efficient boundary element methods for three dimensional exterior viscous flows', *Numer. Methods PDE*, **2**, 273–297 (1986).
5. A. O. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969, pp. 49–64.
6. F. K. G. Odquist, 'Über die Randwertaufgaben der Hydrodynamik zäher Flüssigkeiten', *Math. Z.*, **32**, 201–208 (1930).
7. W. Borchers, 'Über das Anfangsrandwertproblem der instationären Stokes-Gleichung', *Z. Angew. Math. Mech.*, **65**, T329 (1985).
8. M. G. Duffy, 'Quadrature over a pyramid or cube of integrands with a singularity at a vertex', *SIAM J. Numer. Anal.*, **19**, 1260–1262 (1982).